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A critique of supersymmetric Weinberg–Salam models

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Abstract. We write down the Lagrangian of a 'supersymmetric Weinberg–Salam model' which is invariant under the action of the gauge group SU(2/1) and under rotations in a six-dimensional graded space. We show that such a Lagrangian does not, in fact, naturally give rise to the Weinberg–Salam model.

1. Introduction

The Weinberg–Salam model is the currently accepted model of weak and electromagnetic interactions. Conventional attempts to unify it have led to theories containing large numbers of (so far) unobserved particles. Recently Ne'eman (1979), Fairlie (1979 a, b), Dondi and Jarvis (1979a), Squires (1979), Taylor (1979a, b) and Pickup and Taylor (1979) have made 'unconventional' attempts using supersymmetry; we discuss first the motivation behind these schemes, and then the associated problems.

The gauge group used is SU(2/1); this contains the SU(2) × U(1) of the Weinberg–Salam model. However, it also contains a doublet and its conjugate and thus, it was hoped, the Higgs field could be naturally incorporated. The fundamental 3 × 3 representation of the gauge field may be written

$$\left(\begin{array}{c|c} W^\circ, W^\pm & \phi \\ \hline \phi^+ & B^\circ \end{array}\right).$$

Now, the Weinberg–Salam model contains an arbitrary parameter θ_w, the Weinberg angle. Since SU(2/1) models combine the SU(2) and U(1) parts, the Weinberg angle is not a free parameter of these theories, but in fact has been determined to be 30° (Fairlie 1979a, b, Ne'eman 1979), in good agreement with recent experiments. Moreover, in the Weinberg–Salam model the left-handed lepton and its neutrino must be assigned to a doublet, and the right-handed lepton to a singlet. In SU(2/1) models, however, they may all be assigned to the fundamental triplet, which contains an SU(2) doublet and a singlet. This arrangement allows the natural choice of charge matrix (Fairlie 1979a, b). Furthermore there is a four-dimensional representation of SU(2/1) which contains an SU(2) doublet and two singlets; this seems a natural way to introduce left- and right-handed quarks.

So far all seems well, but in fact there are problems even with the incorporation of the Higgs field. Since SU(2/1) is a Lie supergroup, some of its parameters (those other than the parameters of the SU(2) and U(1) subgroups) are elements of a Grassman algebra, i.e. they anticommute with each other. This means that the doublet associated with the odd generators is an anticommuting field, and consequently has the wrong

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spin-statistics. This may be countered by assigning the doublet to a completely separate multiplet (Dondi and Jarvis 1979b), but to incorporate the Higgs particle into the gauge field Dondi and Jarvis (1979a) defined their theory over an extended space–time: to the usual four-dimensional Minkowski space it is necessary to add two extra fermionic (anticommuting) dimensions so that the last two components of the Higgs field are commuting. However, this also means that the last two components of all other fields, in addition to the first four components of the Higgs field, will be anticommuting.

In this paper we shall write down a 'supersymmetric Weinberg–Salam model Lagrangian' (without fermions). By this, we mean a Lagrangian that is a scalar with respect to the six-dimensional space–time supersymmetry mentioned above and which is invariant under the action of SU(2/1). It is the hope of many people that the Weinberg–Salam model will emerge naturally from such a Lagrangian; in fact, we shall show that it does not do so.

This paper is in three parts. In the first part we review SU(2/1) and its representations. In particular we shall discuss the metric of the group which, unlike the metric of the usual gauge groups, is not a simple delta function. We proceed to show how to write down terms which are invariant under the action of the group, in preparation for the construction of the gauge-invariant Lagrangian.

In the second part, we will deal with space–time transformations. We shall consider rotations in the full six-dimensional space. Since the Lagrangian must be a scalar we show how to form rotational invariants from six-dimensional vectors and second-rank tensors.

In the last part, we shall write down our basic fully invariant Lagrangian. We will show that the U(1)-invariant term necessarily has the opposite sign to the SU(2)-invariant part, in contrast to the Weinberg–Salam model and, further, that fields with the wrong spin-statistics are still present. Moreover, the Higgs field does not appear in the straightforward way that one might have hoped. We discuss the spatial dependence of the fields, but point out that freedom of choice in this matter does not seem to help. Finally we show that the measures necessary to extract something like the Weinberg–Salam Lagrangian are so drastic as to render this approach pointless.

2. Review of SU(2/1) and its representations

2.1. The Lie superalgebra

We write the SU(2/1) algebra as (cf Dondi and Jarvis 1979a)

$$\begin{aligned}
&[R_0, R_m] = 0 \\
&[R_i, R_j] = \mathrm{i}\epsilon_{ijk} R_k \\
&[R_m, Q_a] = -\tfrac{1}{2}(\sigma_m)_a^b Q_b \\
&[R_m, \bar{Q}_a] = \tfrac{1}{2}(\sigma_m)_a^{*b} \bar{Q}_b \\
&\{Q_a, \bar{Q}_b\} = \tfrac{1}{2}(\sigma^m)_a^b R_m \\
&\{Q_a, Q_b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0
\end{aligned} \tag{1}$$

where

$$m = 0, 1, 2, 3$$

$$i, j, k = 1, 2, 3$$

$$a, b = 1, 2$$

$$\sigma_m = (\sigma_0, \boldsymbol{\sigma})$$

$$\sigma^m R_m \equiv -\sigma_0 R_0 + \sigma_i R_i.$$

This is a Z_2-graded algebra, i.e. its generators may be divided into two classes, even and odd. In this paper we label these by R and Q, respectively. It can be seen that the R's generate $\mathrm{SU}(2)\times\mathrm{U}(1)$.

In the notation of Ne'eman (1979) a Lie superalgebra can be written

$$[\lambda_A, \lambda_B\} = C^{C}_{AB}\lambda_C \tag{2}$$

where

$$[\,,\} = \begin{cases} [\,,\,] \text{ if } \lambda_A, \lambda_B \text{ both 'even'} \\ [\,,\,] \text{ if one 'even', one 'odd'} \\ \{\,,\} \text{ if both 'odd'.} \end{cases}$$

For SU(2/1) the labels $A, B, C, \ldots$, etc, run over eight values, with the set

$$\lambda_A = R_0, R_i, Q_a, \bar{Q}_a.$$

The algebra (1) may be represented by ordinary 3×3 matrices:

$$R_0 = -\frac{1}{2}\left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & 2 \end{array}\right) \qquad R_1 = \frac{1}{2}\left(\begin{array}{cc|c} & 1 & \\ 1 & & \\ \hline & & \end{array}\right)$$

$$R_2 = \frac{1}{2}\left(\begin{array}{cc|c} & -\mathrm{i} & \\ \mathrm{i} & & \\ \hline & & \end{array}\right) \qquad R_3 = \frac{1}{2}\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & \end{array}\right)$$

$$Q_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|c} & & \\ & & \\ \hline 1 & 0 & \end{array}\right) \qquad Q_2 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|c} & & \\ & & \\ \hline 0 & 1 & \end{array}\right) \tag{3}$$

$$\bar{Q}_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|c} & & 1 \\ & & 0 \\ \hline & & \end{array}\right) \qquad \bar{Q}_2 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|c} & & 0 \\ & & 1 \\ \hline & & \end{array}\right)$$

The adjoint representation is defined by

$$(\lambda_A)^{C}_{B} = C^{B}_{AC} \tag{4}$$

where B and C are the row and column labels, respectively. (One can also use $(\lambda_A)^{C}_{B} = -(-)^{AB} C^{C}_{A\,B}$ which gives the same metric.)

It can be shown that this is indeed a representation by using the 'graded Jacobi identity' (see, for example, Corwin *et al* 1975):

$$[\lambda_A, [\lambda_B, \lambda_C\}\} = [[\lambda_A, \lambda_B\}, \lambda_C\} + (-)^{AB}[\lambda_B, [\lambda_A, \lambda_C\}\} \tag{5}$$

where

$$(-)^{AB}=\begin{cases}+1 \text{ if } \lambda_A, \lambda_B \text{ both even}\\ +1 \text{ if one even, one odd}\\ -1 \text{ if both odd.}\end{cases}$$

Substituting the structure constants and using the linear independence of the generators yields (2).

Explicitly, the adjoint representation is

$$R_0=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\\hline 0&0&0&0&-1&0&0&0\\0&0&0&0&0&-1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{array}\right)\qquad R_1=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&-2i&0&0&0&0\\0&0&2i&0&0&0&0&0\\\hline 0&0&0&0&0&-1&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{array}\right)$$

$$R_2=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&0&0&0&0\\0&0&0&2i&0&0&0&0\\0&0&0&0&0&0&0&0\\0&-2i&0&0&0&0&0&0\\\hline 0&0&0&0&0&-i&0&0\\0&0&0&0&i&0&0&0\\0&0&0&0&0&0&0&-i\\0&0&0&0&0&0&i&0\end{array}\right)\qquad R_3=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&0&0&0&0\\0&0&-2i&0&0&0&0&0\\0&2i&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\\hline 0&0&0&0&-1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&-1\end{array}\right)$$

$$Q_1=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&-i\\0&0&0&0&0&0&1&0\\\hline 1&0&0&1&0&0&0&0\\0&1&-i&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{array}\right)\qquad Q_2=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&i&0\\0&0&0&0&0&0&0&-1\\\hline 0&1&i&0&0&0&0&0\\1&0&0&-1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{array}\right)\tag{6}$$

$$\bar{Q}_1=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&-1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&i&0&0\\0&0&0&0&1&0&0&0\\\hline 0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\-1&0&0&-1&0&0&0&0\\0&-1&-i&0&0&0&0&0\end{array}\right)\qquad \bar{Q}_2=\frac{1}{2}\left(\begin{array}{cccc|cccc}0&0&0&0&0&-1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&-i&0&0&0\\0&0&0&0&0&-1&0&0\\\hline 0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&-1&i&0&0&0&0&0\\-1&0&0&1&0&0&0&0\end{array}\right)$$

2.2. *The group SU(2/1)*

An element of the group may be written in the form

$$g = \exp[\mathrm{i}(\theta_0 R_0 + \theta_i R_i + \theta_a Q_a + \bar{Q}_a \bar{\theta}_a)]. \tag{7}$$

The parameters θ are graded, i.e. the set (θ_0, θ_i) are (real) numbers and the set $(\theta_a, \bar{\theta}_a)$ are elements of a Grassman algebra:

$$\{\theta_a, \theta_b\} = \{\bar{\theta}_a, \bar{\theta}_b\} = \{\theta_a, \bar{\theta}_b\} = 0.$$

Notice that in the fundamental representation (3) the R's are Hermitian and the Q's are the Hermitian conjugates of the $\bar{Q}$'s. This is necessary for SU(2/1) to be unitary in its fundamental representation. Infinitesimally,

$$g^+ g = 1 + \mathrm{i}(\theta_0 R_0 - \theta_0^* R_0^+ + \theta_i R_i - \theta_i^* R_i^+ + \theta_a Q_a - Q_a^+ \bar{\theta}_a + \bar{Q}_a \bar{\theta}_a - \bar{\theta}_a \bar{Q}_a^+).$$

Reality and Hermiticity ensure that the first four terms vanish, and the last four vanish since $Q_a^+ = \bar{Q}_a$ in the fundamental representation.

A group element of SU(2/1) can be written in the form

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{8}$$

where the elements of A and D are just numbers, and the elements of B and C are anticommuting numbers. Any matrix with this structure will be referred to as 'correctly graded'. It is easily seen that this graded structure is preserved under multiplication.

The graded trace or supertrace is defined by

$$\mathrm{str}(g) = \mathrm{tr}(A) - \mathrm{tr}(D). \tag{9}$$

By multiplying group elements component by component we see that the supertrace of group elements is cyclic:

$$\mathrm{str}(gg') = \mathrm{str}(g'g). \tag{10}$$

Note that this is not true of the generators of the group since they are not correctly graded matrices:

$$\mathrm{str}(\lambda_A \lambda_B) = (-)^{AB}\,\mathrm{str}(\lambda_B \lambda_A) \tag{11}$$

$(\mathrm{str}(C^C_{AB}\lambda_C) = 0$ because the generators are supertraceless).

2.3. *The metric of SU(2/1)*

The Killing metric is defined by

$$g_{AB} = \mathrm{str}((\mathrm{adj}\,\lambda_A)(\mathrm{adj}\,\lambda_B)) \tag{12}$$

and may be used to raise and lower indices in the usual way. If we define λ^B implicitly by

$$\lambda_A = g_{AB}\lambda^B$$

and g^{AB} by

$$\lambda^A = g^{AB}\lambda_B$$

then

$$g_{AC}g^{CB} = g^{BC}g_{CA} = \delta^B_A.$$

We point out that

$$g^{AB} = \mathrm{str}((\mathrm{adj}\,\lambda^B)(\mathrm{adj}\,\lambda^A)).$$

Explicitly

$$g_{AB} = \left(\begin{array}{cccc|cccc} -1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \hline & & & & & & -1 & 0 \\ & & & & & & 0 & -1 \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \end{array}\right). \tag{13}$$

The fact that the metric has both symmetric and antisymmetric parts causes some trouble—we point out that in general, for any quantities X^A,

$$X^A X_A \neq X_A X^A.$$

2.4. Invariants under SU(2/1)

We recall first some features of ordinary Yang–Mills theories.

Consider a field ϕ which transforms like

$$\phi \to g\phi \tag{14}$$

under a local gauge transformation. (We do not distinguish between an element of a group and its representation.) Now since $g = g(x)$, the ordinary derivative of ϕ will not transform in the same way. To overcome this, we define the covariant derivative

$$\mathrm{D}^\mu = \partial^\mu + \mathrm{i}eA^\mu_a T^a \tag{15}$$

where the T^a are the generators of the (Lie) gauge group, and $A^\mu_a T^a \equiv \boldsymbol{A}^\mu(x)$ is the gauge field, transforming like

$$\boldsymbol{A}^\mu \to g\boldsymbol{A}^\mu g^{-1} + \frac{\mathrm{i}}{e}(\partial^\mu g)g^{-1}. \tag{16}$$

We then have

$$\mathrm{D}^\mu\phi \to g\mathrm{D}^\mu\phi. \tag{17}$$

If we define $F^{\mu\nu}_a$ by

$$\begin{aligned} F^{\mu\nu}_a T^a\phi &= \frac{1}{\mathrm{i}e}[\mathrm{D}^\mu, \mathrm{D}^\nu]\phi \\ &= (\partial^\mu \boldsymbol{A}^\nu - \partial^\nu \boldsymbol{A}^\mu + \mathrm{i}e[\boldsymbol{A}^\mu, \boldsymbol{A}^\nu])\phi \end{aligned} \tag{18}$$

then we see immediately that under a gauge transformation

$$F_a^{\mu\nu}T^a \to gF_a^{\mu\nu}T^a g^{-1}. \tag{19}$$

Now since

$$\mathrm{tr}(T^aT^b) \propto \delta^{ab} \tag{20}$$

we have (ignoring the proportionality constant)

$$\begin{aligned} F^{\mu\nu}{}_a F_{\mu\nu a} &= F^{\mu\nu}{}_a \delta^{ab} F_{\mu\nu b} \\ &= F^{\mu\nu}{}_a\, \mathrm{tr}(T^aT^b) F_{\mu\nu b} \\ &= \mathrm{tr}(F^{\mu\nu}{}_a T^a F_{\mu\nu b} T^b). \end{aligned} \tag{21}$$

Thus under a gauge transformation

$$\begin{aligned} F^{\mu\nu}{}_a F_{\mu\nu a} &\to \mathrm{tr}(gF^{\mu\nu}{}_a T^a g^{-1} g F_{\mu\nu b} T^b g^{-1}) \\ &= F^{\mu\nu}{}_a F_{\mu\nu a} \end{aligned}$$

by the cyclic property of the trace.

Thus $F^{\mu\nu}{}_a F_{\mu\nu a}$ is an invariant of Yang–Mills theories. It will be the object of this section to determine the corresponding invariant for SU(2/1).

In the final section of this paper, when we come to evaluate the gauge field commutators (and anticommutators), we shall discover that, in order to stay within the algebra, all the 'odd' components of vectors, tensors, fields, etc, must anticommute with the odd generators (as one would expect from their grading). This means that care will be necessary when summing the components of a field over the generators of the group. We shall define

$$\begin{aligned} F^A\lambda_A &\equiv F_0R_0 + F_iR_i + F_aQ_a + \bar{Q}_a\bar{F}_a \\ &= F^0R^0 + F^iR^i + \bar{F}^a\bar{Q}^a + Q^aF^a. \end{aligned} \tag{22}$$

Then the SU(2/1) analogue of (18) (in four dimensions) is defined to be

$$F_{\mu\nu}^A\lambda_A\phi = \frac{1}{\mathrm{i}e}[\mathrm{D}_\mu, \mathrm{D}_\nu]\phi$$

where

$$\mathrm{D}_\mu = \partial_\mu + \mathrm{i}eA_\mu{}^A\lambda_A.$$

Consequently we know that (ignoring space–time indices)

$$F^A\lambda_A \to gF^A\lambda_A g^{-1}.$$

Now consider (taking the generators to be in the adjoint representation)

$$\begin{aligned} &\mathrm{str}(F^A\lambda_AF^B\lambda_B) \\ &\quad = \mathrm{str}(F_0R_0F_0R_0) + \mathrm{str}(F_iR_iF_jR_j) + \mathrm{str}(F_aQ_a\bar{Q}_b\bar{F}_b) + \mathrm{str}(\bar{Q}_a\bar{F}_aF_bQ_b) \\ &\quad = -F_0^2 + F_i^2 - F_a\bar{F}_a + \bar{F}_aF_a. \end{aligned} \tag{23}$$

We would expect to be able to commute λ_B through F^B in order to write this analogously to $F^{\mu\nu}{}_a\delta^{ab}F_{\mu\nu b}$, and if we are careful with signs and definitions we can indeed do so. But when we extend the theory to six dimensions the space–time indices on F (which we have so far ignored) will be graded, and this will make things more complicated. In view of this it seems simplest always to write invariants in terms of supertraces.

The graded space–time indices on F also mean that, for some values of these indices, $F^A\lambda_A$ will be a wrongly graded matrix (i.e. it will be even where a correctly graded matrix is odd, and vice versa). However, the space–time metric is box-diagonal (see § 3) and if we use this to sum over the space–time indices then this will ensure that $F^A\lambda_A F^B\lambda_B$ is correctly graded, since the product of two wrongly graded matrices is correctly graded.

It follows that under a gauge transformation

$$\begin{aligned}\operatorname{str}(F^A\lambda_A F^B\lambda_B) &\rightarrow \operatorname{str}(gF^A\lambda_A F^B\lambda_B g^{-1})\\ &= \operatorname{str}(F^A\lambda_A F^B\lambda_B)\end{aligned}$$

irrespective of any suppressed indices on F, and thus this is the desired SU(2/1) invariant.

3. Space–time transformations

As we pointed out in the introduction, Dondi and Jarvis (1979a) have set up supersymmetric Weinberg–Salam models on graded manifolds—i.e. manifolds some of whose coordinates anticommute—so as to provide a Higgs field with the correct spin-statistics. We stress that this technique not only does not remove the Higgs field with the wrong spin-statistics, but actually introduces wrong spin-statistic components into ordinary fields.

We shall treat a six-dimensional space. The relevant metric is

$$g_{pq}=\left(\begin{array}{cccc|cc} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ \hline & & & & 0 & 1 \\ & & & & -1 & 0 \end{array}\right). \tag{24}$$

We use

$$p, q, r, s, \text{etc} = \begin{cases}\mu, \nu, \rho, \text{etc} = 0, 1, 2, 3\\ \alpha, \beta, \gamma, \text{etc} = 5, 6.\end{cases}$$

Since g_{pq} is neither symmetric nor antisymmetric we shall have exactly the same problems as with g_{AB}.

Define

$$z^p = (x^\mu, \xi^1, \xi^2)$$

and z_p by

$$z_p = g_{pq} z^q = (x_\mu, \xi^2, -\xi^1)$$

and g^{pq} by

$$z^p = g^{pq} z_q.$$

This gives

$$g^{pq}g_{qr} = \delta^p_r = g_{pq}g^{qr}.$$

We obtain (numerically)

$$g^{pq} = g_{qp} \qquad g_p{}^q = \delta^q_p \qquad g^p{}_q = (-)^p\delta^p_q.$$

We require our Lagrangian to transform as a scalar under OSp(3, 1/2), which is the full group of rotations in this six-dimensional space. Define the transformation of z_p by

$$z'_p = a_p{}^q z_q. \tag{25}$$

Then multiplying by g^{rp} we have

$$g^{rp}z'_p = z'^r = g^{rp}a_p{}^q g_{qs}z^s.$$

Hence, if we define

$$z'^p = b^p{}_q z^q \tag{26}$$

then

$$\begin{aligned} b^p{}_q &= g^{pr}a_r{}^s g_{sq} \\ &= a^{ps}g_{sq}. \end{aligned}$$

As a check we take the identity transformation on z_p, when

$$a_p{}^q = g_p{}^q = \delta^q_p$$

and then

$$\begin{aligned} b^p{}_q &= g^{pr}\delta^s_r g_{sq} \\ &= \delta^p_q \qquad \text{also.} \end{aligned}$$

For an infinitesimal transformation we define

$$a_p{}^q = g_p{}^q + \Delta\omega_p{}^q. \tag{27}$$

This gives

$$b^p{}_q = \delta^p_q + \Delta\omega^{pr}g_{rq}. \tag{28}$$

Then

$$\begin{aligned} z^{p'}z'_p &= b^p{}_q z^q a_p{}^r z_r \\ &= z^p z_p + \Delta\omega^{ps}g_{sq}z^q\delta^r_p z_r + \delta^p_q z^q \Delta\omega_p{}^r z_r. \end{aligned}$$

For $z^p z_p$ to be invariant under rotations we require

$$\Delta\omega^{pq}z_q z_p + z^p\Delta\omega_p{}^q z_q = 0. \tag{29}$$

Expanding the summations into sums over odd and even coordinates, and remembering the grading of z and $\Delta\omega$, and the fact that $\Delta\omega^{\mu\alpha}z_\alpha = -\Delta\omega^\mu{}_\alpha z^\alpha$ we obtain the conditions

$$\begin{aligned} \Delta\omega^{\mu\nu} &= -\Delta\omega^{\nu\mu} \qquad \text{(as expected)} \\ \Delta\omega^{\mu\alpha} &= -\Delta\omega^{\alpha\mu} \\ \Delta\omega^{\alpha\beta} &= \Delta\omega^{\beta\alpha}. \end{aligned} \tag{30}$$

It is easily seen that $z_p z^p$ is not invariant under these conditions. We therefore conclude that the invariant scalar product is

$$z^p z_p = x^\mu x_\mu + \xi^1\xi^2 - \xi^2\xi^1$$
$$= z^p g_{pq} z^q = z_p g^{qp} z_q.$$

For completeness we derive the algebra of OSp(3, 1/2). We write $\Delta\omega_p{}^q$ in terms of the parameters and generators of the group

$$\Delta\omega_p{}^q = \Delta\phi^{\mu\nu}(J_{\mu\nu})_p{}^q + \Delta\phi^{\mu\alpha}(\Pi_{\mu\alpha})_p{}^q + \Delta\phi^{\alpha\beta}(\Xi_{\alpha\beta})_p{}^q.$$

The parameters $\Delta\phi^{pq}$ have the same grading and symmetry as $\Delta\omega^{pq}$; there are seventeen of them since the independent generators consist of six J, eight Π and three Ξ.

To determine the symmetry of the generators as matrices, we need to know the relation between $\Delta\omega_p{}^q$ and $\Delta\omega_q{}^p$. From

$$\Delta\omega_p{}^q = -(-)^{pq}\Delta\omega^q{}_p$$

and

$$\Delta\omega^q{}_p = g^{qr} g_{ps}\Delta\omega_r{}^s$$

we obtain

$$\Delta\omega_p{}^q = -(-)^{pq} g^{qr} g_{ps}\Delta\omega_r{}^s. \tag{31}$$

Substituting particular values of p, q, r and s we obtain the following relations:

$$\begin{aligned}
&\Delta\omega_0{}^i = \Delta\omega_i{}^0\\
&\Delta\omega_i{}^j = -\Delta\omega_j{}^i\\
&\Delta\omega_0{}^\alpha = -g^{\alpha\beta}\Delta\omega_\beta{}^0\\
&\Delta\omega_i{}^\alpha = g^{\alpha\beta}\Delta\omega_\beta{}^i\\
&\Delta\omega_5{}^5 = -\Delta\omega_6{}^6\\
&\Delta\omega_5{}^6,\ \Delta\omega_6{}^5 \text{ independent.}
\end{aligned} \tag{32}$$

Then we may write the components of the generators (in this 6×6 representation) as

$$\begin{aligned}
&(J_{\mu\nu})_p{}^q = \mathrm{i}(\delta_{\mu p} g_{\nu q} - g_{\mu q}\delta_{\nu p})\\
&(\Pi_{\mu\alpha})_p{}^q = \mathrm{i}(g_{\mu q} g_{\alpha p} + \delta_{\mu p}\delta_{\alpha q})\\
&(\Xi_{\alpha\beta})_p{}^q = (g_{p\alpha}\delta_{\beta q} + g_{p\beta}\delta_{\alpha q}).
\end{aligned} \tag{33}$$

They give rise to the following algebra:

$$\begin{aligned}
&[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho})\\
&[\Xi_{\alpha\beta}, \Xi_{\gamma\delta}] = g_{\alpha\delta}\Xi_{\beta\gamma} + g_{\beta\gamma}\Xi_{\alpha\delta} + g_{\alpha\gamma}\Xi_{\beta\delta} + g_{\beta\delta}\Xi_{\alpha\gamma}\\
&\{\Pi_{\mu\alpha}, \Pi_{\nu\beta}\} = g_{\mu\nu}\Xi_{\alpha\beta} - \mathrm{i}g_{\alpha\beta}J_{\mu\nu}\\
&[J_{\mu\nu}, \Pi_{\rho\alpha}] = \mathrm{i}(g_{\nu\rho}\Pi_{\mu\alpha} - g_{\mu\rho}\Pi_{\nu\alpha})\\
&[\Xi_{\alpha\beta}, \Pi_{\mu\gamma}] = g_{\beta\gamma}\Pi_{\mu\alpha} + g_{\alpha\gamma}\Pi_{\mu\beta}\\
&[J_{\mu\nu}, \Xi_{\alpha\rho}] = 0.
\end{aligned} \tag{34}$$

Dondi and Jarvis (1979a) have given a differential representation of this algebra.

3.1. Gauge theories on graded manifolds

We have already discussed the implications of using SU(2/1) as a gauge group, and have seen that $\mathrm{str}(F^A\lambda_A F^B\lambda_B)$ is invariant under the action of the group. We complete the discussion by extending our theory to six graded dimensions.

The covariant derivative is

$$\mathrm{D}_p = \partial_p + \mathrm{i}eA_p^A\lambda_A$$

where

$$A_p^A\lambda_A = \boldsymbol{A}_p(x, \xi)$$

and transforms in the usual way.

Using the fact that ∂_p is a 'graded-Leibniz' operator, i.e. that

$$\partial_p(z^q z_q) = (\partial_p z^q)z_q + (-)^{pq} z^q(\partial_p z_q)$$

it is then easily shown that

$$\mathrm{D}_p\phi \to g\mathrm{D}_p\phi$$

if

$$\phi \to g\phi.$$

We define the second-rank tensor by

$$[\mathrm{D}_p, \mathrm{D}_q\} = \mathrm{i}eF_{pq}^A\lambda_A$$

and then

$$F_{pq}^A\lambda_A = \partial_p\boldsymbol{A}_q - (-)^{pq}\partial_q\boldsymbol{A}_p + \mathrm{i}e[\boldsymbol{A}_p, \boldsymbol{A}_q\}.$$

We must now form a term bilinear in F which is invariant under both SU(2/1) and OSp(3, 1/2); the analogue of the $F^{\mu\nu a}F_{\mu\nu a}$ term in Yang–Mills theories. In fact, the required term is $\mathrm{str}(F^{pqA}\lambda_A F_{qp}^B\lambda_B)$: that this is a scalar under OSp(3, 1/2) seems plausible since $z^p z^q z_q z_p$ obviously is. We now outline the proof.

$$\begin{aligned}\mathrm{str}(F^{pqA}\lambda_A F_{qp}^B\lambda_B) &= -\frac{1}{e^2}\mathrm{str}([\mathrm{D}^p, \mathrm{D}^q\}[\mathrm{D}_q, \mathrm{D}_p\}) \\ &\to -\frac{1}{e^2}\mathrm{str}([b^p{}_r\mathrm{D}^r, b^q{}_s\mathrm{D}^s\}[a_q{}^t\mathrm{D}_t, a_p{}^u\mathrm{D}_u\}).\end{aligned} \tag{35}$$

The delta functions in the transformations a and b just give back the original term, and to first order we are left with

$$\begin{aligned}-\frac{1}{e^2}\mathrm{str}(([\Delta\omega^{ps}g_{sr}\mathrm{D}^r, \mathrm{D}^q\} + [\mathrm{D}^p, \Delta\omega^{qr}g_{rs}\mathrm{D}^s\})[\mathrm{D}_q, \mathrm{D}_p\} \\ + [\mathrm{D}^p, \mathrm{D}^q\}([\Delta\omega_q{}^t\mathrm{D}_t, \mathrm{D}_p\} + [D_q, \Delta\omega_p{}^u\mathrm{D}_u\})).\end{aligned} \tag{36}$$

After some algebra it can be shown that the two sets of terms in the supertrace cancel pair-wise, and thus $\mathrm{str}(F^{pqA}\lambda_A F_{qp}^A\lambda_B)$ is the relevant invariant.

4. The Lagrangian

As a basic Lagrangian, without fermions, we take

$$\begin{aligned}\mathscr{L} &= \tfrac{1}{4}\,\mathrm{str}(F^{pqA}\lambda_A F^{B}_{qp}\lambda_B)\\ &= -\tfrac{1}{4}F^{\mu\nu m}F_{\mu\nu m} - \tfrac{1}{2}F^{\mu\alpha m}F_{\mu\alpha m} + \tfrac{1}{4}F^{\alpha\beta m}F_{\alpha\beta m} - \tfrac{1}{2}\bar{F}^{\mu\nu}{}_{a}F_{\mu\nu a}\\ &\quad - \bar{F}^{\mu\alpha}{}_{a}F_{\mu\alpha a} + \tfrac{1}{2}\bar{F}^{\alpha\beta}{}_{a}F_{\alpha\beta a}.\end{aligned} \tag{37}$$

Let us consider the first term. Since it is obviously invariant under $SU(2)\times U(1)$ one might hope that at least this term reproduced part of the Weinberg–Salam Lagrangian. But (from (13))

$$-\tfrac{1}{4}F^{\mu\nu m}F_{\mu\nu m} = \tfrac{1}{4}F^{\mu\nu 0}F^{0}_{\mu\nu} - \tfrac{1}{4}\boldsymbol{F}^{\mu\nu}\,.\,\boldsymbol{F}_{\mu\nu}. \tag{38}$$

In the Weinberg–Salam model these two terms have the same sign. The sign difference here is not trivial—it arises ultimately from the fact that the generators of SU(2/1) must be supertraceless.

We must also consider the other terms in the Lagrangian. In order to see what is happening with the fields we evaluate the gauge field (anti)commutators. It is here that we have to assume that the odd-field components anticommute with the odd generators if we are to stay within the algebra. We obtain

$$\begin{aligned}[\boldsymbol{A}^{\mu}, \boldsymbol{A}^{\nu}] &= \tfrac{1}{2}(A^{\mu}_{a}\bar{A}^{\nu}_{b} + \bar{A}^{\mu}_{b}A^{\nu}_{a})(\sigma^{m})^{b}_{a}R_m + \mathrm{i}A^{\mu}_{i}A^{\nu}_{j}\epsilon_{ijk}R_k + \tfrac{1}{2}(A^{\mu}_{a}A^{\nu}_{0} - A^{\mu}_{0}A^{\nu}_{a})(\sigma_0)^{b}_{a}Q_b\\ &\quad + \tfrac{1}{2}(A^{\mu}_{a}A^{\nu}_{i} - A^{\mu}_{i}A^{\nu}_{a})(\sigma_i)^{b}_{a}Q_b + \tfrac{1}{2}(\bar{A}^{\mu}_{a}A^{\nu}_{0} - A^{\mu}_{0}\bar{A}^{\nu}_{a})(\sigma^{*}_{0})^{b}_{a}\bar{Q}_b\\ &\quad + \tfrac{1}{2}(\bar{A}^{\mu}_{a}A^{\nu}_{i} - A^{\mu}_{i}\bar{A}^{\nu}_{a})(\sigma^{*}_{i})^{b}_{a}\bar{Q}_b\\ [\boldsymbol{A}^{\mu}, \boldsymbol{A}^{\alpha}] &= -\tfrac{1}{2}(A^{\mu}_{a}\bar{A}^{\alpha}_{b} + \bar{A}^{\mu}_{b}A^{\alpha}_{a})(\sigma^{m})^{b}_{a}R_m + \mathrm{i}A^{\mu}_{i}A^{\alpha}_{j}\epsilon_{ijk}R_k - \tfrac{1}{2}(A^{\mu}_{a}A^{\alpha}_{0} + A^{\mu}_{0}A^{\alpha}_{a})(\sigma_0)^{b}_{a}Q_b\\ &\quad - \tfrac{1}{2}(A^{\mu}_{a}A^{\alpha}_{i} + A^{\mu}_{i}A^{\alpha}_{a})(\sigma_i)^{b}_{a}Q_b - \tfrac{1}{2}(\bar{A}^{\mu}_{a}A^{\alpha}_{0} + \bar{A}^{\alpha}_{a}A^{\mu}_{0})(\sigma^{*}_{0})^{b}_{a}\bar{Q}_b\\ &\quad - \tfrac{1}{2}(\bar{A}^{\mu}_{a}A^{\alpha}_{i} + A^{\mu}_{i}\bar{A}^{\alpha}_{a})(\sigma^{*}_{i})^{b}_{a}\bar{Q}_b\\ \{\boldsymbol{A}^{\alpha}, \boldsymbol{A}^{\beta}\} &= -\tfrac{1}{2}(\bar{A}^{\alpha}_{b}A^{\beta}_{a} + A^{\alpha}_{a}\bar{A}^{\beta}_{b})(\sigma^{m})^{b}_{a}R_m + \mathrm{i}A^{\alpha}_{i}A^{\beta}_{j}\epsilon_{ijk}R_k\\ &\quad - \tfrac{1}{2}(A^{\alpha}_{a}A^{\beta}_{0} + A^{\alpha}_{0}A^{\beta}_{a})(\sigma_0)^{b}_{a}Q_b - \tfrac{1}{2}(A^{\alpha}_{a}A^{\beta}_{i} + A^{\alpha}_{i}A^{\beta}_{a})(\sigma_i)^{b}_{a}Q_b\\ &\quad - \tfrac{1}{2}(\bar{A}^{\alpha}_{a}A^{\beta}_{0} + A^{\alpha}_{0}\bar{A}^{\beta}_{a})(\sigma^{*}_{0})^{b}_{a}\bar{Q}_b - \tfrac{1}{2}(\bar{A}^{\alpha}_{a}A^{\beta}_{i} + A^{\alpha}_{i}\bar{A}^{\beta}_{a})(\sigma^{*}_{i})^{b}_{a}\bar{Q}_b.\end{aligned}$$

It is obvious that the theory is inundated with wrong spin-statistic fields. They arise because we have used a graded Lie gauge group and because the theory is formulated on a graded manifold. In a full quantum-mechanical treatment it is not possible to set the unwanted field components equal to zero.

One might hope to dispose of the wrong spin-statistic fields by choosing some particular spatial dependence and integrating over the anticommuting dimensions, using

$$\int \mathrm{d}a = 0$$

$$\int a\,\mathrm{d}a = 1$$

where a is an element of a Grassman algebra. This would break the OSp(3, 1/2) invariance down to the required Lorentz invariance. However, it would be necessary to dispose of the components of the W and B fields associated with the odd generators of

SU(2/1) and the components of the Higgs field associated with the even generators, and we have not been able to find a spatial dependence to do this.

Finally, we point out that it is possible to extract something like the Weinberg–Salam model if we take the trace

$$\mathrm{tr}(F^{pqA}\lambda_A F^B_{qp}\lambda_B) = F^{pq}{}_0 F_{qp0} + 3F^{pq}{}_i F_{qpi}.$$

By restricting the theory to four dimensions and rescaling the Abelian part we can indeed reproduce the first two terms of the Weinberg–Salam model (although the Higgs field has disappeared). But this is not a supersymmetric model at all; by taking the trace we are breaking the SU(2/1) invariance down to SU(2) × U(1), and if we then restrict the theory to four dimensions we are doing no more than Weinberg did in the first place.

To conclude then—in this paper we have established a consistent formalism and have written down the full SU(2/1) and OSp(3, 1/2) invariant Lagrangian. This makes clear the difficulties involved in attempts to establish a realistic physical model. Indeed the difficulties seem such as to make it unlikely that a realistic physical model will easily be established.

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References

Corwin L, Ne'eman Y and Sternberg S 1975 *Rev. Mod. Phys.* **47** 573
Dondi P H and Jarvis P D 1979a *Phys. Lett.* **84B** 75
—— 1979b *Coseners House Meeting, May 12–13th*
Fairlie D B 1979a *J. Phys. G: Nucl. Phys.* **5** L55
—— 1979b *Phys. Lett.* **82B** 97
Ne'eman Y 1979 *Phys. Lett.* **81B** 190
Pickup C and Taylor J G 1979 *Preprint* King's College, London
Squires E J 1979 *Phys. Lett.* **82B** 395
Taylor J G 1979a *Phys. Lett.* **83B** 331
—— 1979b *Preprint* King's College, London